ADIFOR Working Note #7:

Extending Compile-Time Reverse Mode and Exploiting Partial Separability in ADIFOR

by

Christian H. Bischof and Moe EL-Khadiri

Argonne Technical Memorandum MCS-TM-163, 1992

Abstract. The numerical methods employed in the solution of many scientific computing problems require the computation of the gradient of a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \). ADIFOR is a source translator that, given a collection of subroutines to compute \( f \), generates Fortran 77 code for computing the derivative of this function. Using the so-called torsion problem from the MINPACK-2 test collection as an example, this paper explores two issues in automatic differentiation: the efficient computation of derivatives for partial separable functions and the use of the compile-time reverse mode for the generation of derivatives. We show that orders of magnitude of improvement are possible when exploiting partial separability and maximizing use of the reverse mode.

1 Introduction

Differentiation is one of the most fundamental mathematical concepts. In system analysis and control, the investigation into the effect of a disturbance or a change in design parameters on the performance of the overall system is essential. Mathematically, the change can be modeled by the derivative of the system output with respect to a design parameter. Another application is the numerical solution of initial value problems in stiff ordinary differential equations (see, for example [7, 18]). Methods such as implicit Runge-Kutta and backward differentiation formula (BDF) methods require a Jacobian which is either supplied by the user or approximated by finite differences. In the context of optimization, one needs the derivatives of the objective function. For example, given a function

\[ f : \mathbb{R}^n \rightarrow \mathbb{R}, \]

one can find a minimizer \( x_* \) of \( f \) using variable metric methods that involve the iteration

\[
\text{for } i = 1, 2, \ldots \text{ do} \\
\quad \text{Solve } B_i s_i = -\nabla f(x_i) \\
\quad x_{i+1} = x_i + \alpha_i s_i \\
\text{end for}
\]

where \( B_i \) is a symmetric positive definite matrix approximating the Hessian of \( f \) at \( x_i \).

These methods are examples of typical methods applied in numerical computations, where the computation of the derivative is a crucial step in the numerical solution process ([6, 9, 12, 8, 21]). One particular optimization problem is the elastic plastic torsion problem, which arises from the determination of the stress field on an infinitely long cylindrical bar. The infinite-dimensional version of this problem is of the form

\[
\min \{ g(v) : v \in K \},
\]
where \( q : K \to R \) is the quadratic

\[
q(v) = \frac{1}{2} \int_D \| \nabla v(x) \|^2 \, dx - c \int_D v(x) \, dx
\]

for some constant \( c \), and \( D \) is a bounded domain with smooth boundary. The convex set \( K \) is defined by

\[
K = \{ v \in H^1(D) : \| v \| \leq \text{dist}(x, \partial D), x \in D \},
\]

where \( \text{dist}(x, \partial D) \) is the distance function to the boundary of \( D \), and \( H^1(D) \) is the Hilbert space of all functions with compact support in \( D \) such that \( v \) and \( \| \nabla v \|^2 \) belong to \( L^2(D) \). This formulation and the physical interpretation of the torsion problem are discussed in the test problem collection of MINPACK-2 [1]. A finite element approximation of the torsion problem leads to

\[
q(v) = \frac{1}{2} \sum q^L_{i,j}(v) + \frac{1}{2} \sum q^U_{i,j}(v) - h_x h_y \sum w_l(z_{i,j}) v_{i,j},
\]

where

\[
q^L_{i,j}(v) = \mu_{i,j} \left\{ \left( \frac{v_{i+1,j} - v_{i,j}}{h_x} \right)^2 + \left( \frac{v_{i,j+1} - v_{i,j}}{h_y} \right)^2 \right\},
\]

\[
q^U_{i,j}(v) = \lambda_{i,j} \left\{ \left( \frac{v_{i-1,j} - v_{i,j}}{h_x} \right)^2 + \left( \frac{v_{i,j-1} - v_{i,j}}{h_y} \right)^2 \right\},
\]

and \( \mu_{i,j}, \lambda_{i,j}, h_x, \) and \( h_y \) are constants.

Note that \( q^L_{i,j}(v) \) and \( q^U_{i,j}(v) \) are quadratics which depend only on \( v_{i+1,j}, v_{i,j+1}, v_{i,j} \), and on \( v_{i-1,j}, v_{i,j-1}, v_{i,j} \), respectively. The third contribution to \( q(v) \), which is the linear part, depends only on \( v_{i,j} \). So if we define

\[
S_1 = \sum q^L_{i,j}(v),
S_2 = \sum q^U_{i,j}(v),
S_3 = \sum w_l(z_{i,j}) v_{i,j},
\]

then

\[
f(x) = \frac{1}{2} S_1 + \frac{1}{2} S_2 - h_x h_y S_3.
\]

In the MINPACK-2 code for the torsion problem shown in Appendix A, LOOP1, LOOP2, and LOOP3 correspond to the computation of \( S_1 \), \( S_2 \), and \( S_3 \), respectively.

The torsion problem is a particular instance of a particular class of functions that arises often in optimization contexts, the so-called partially separable functions [11,17,19]. These are functions \( f : R^n \to R \) which can be expressed as

\[
f(x) = \sum_{i=1}^{n} a_i f_i(x).
\]
Usually each \( f_i \) depends on only a few (say, \( n_i \)) of the \( x \)'s, and one can take advantage of this fact in computing the (sparse) Hessian of \( f \).

ADIFOR (Automatic Differentiation of Fortran) is a source translator that augments Fortran codes with statements for the computation of derivatives [3, 2]. ADIFOR employs a mixed forward/reverse mode paradigm. The forward mode propagates derivatives of intermediate variables with respect to the input variables; the reverse mode propagates derivatives of the final values with respect to intermediate variables [14]. The forward mode follows the flow of execution of the original program, whereas the reverse mode of automatic differentiation requires the ability to access values generated in the execution of a program in reverse order, which is usually achieved by logging all values on a so-called tape, and then interpreting the tape in reverse order [14, 16, 15]. ADIFOR pioneered the use of the compile-time reverse mode where, instead of logging values at run time, we apply the reverse mode at compile time, thereby eliminating the storage requirements and run-time overhead of the tape scheme.

In this paper, we are concerned with the efficient generation of derivative code through the reverse mode of automatic differentiation, and the efficient use of the generated derivative code for computing gradients of partially separable functions. We use the torsion problem as a case study and explore how to improve the current ADIFOR-generated code and decrease the time and storage complexity of computing derivatives.

The paper is structured as follows. In the next section, we recall the key points about the method that is currently used in ADIFOR to generate derivatives. In Section 3, we then illustrate extensions of the compile-time reverse mode from basic blocks all the way to generating an adjoint code for the whole program. In Section 4, we explore the use of partial separability in computing derivatives. We present experimental results on Sparc-2 and IBM RS6000/550 workstations in Section 5.

2 Current ADIFOR Strategy

Automatic differentiation techniques rely on the fact that every function, no matter how complicated, is executed on a computer as a (potentially very long) sequence of elementary operations such as additions, multiplications, and elementary functions such as sin and cos. By applying the chain rule

\[
\frac{\partial}{\partial t} f(g(t))|_{t=t_0} = \left( \frac{\partial}{\partial s} f(s) \right)_{s=g(t_0)} \left( \frac{\partial}{\partial t} g(t) \right)_{t=t_0}
\]  

over and over again to the composition of those elementary operations, one can compute derivative information of \( f \) exactly and in a completely mechanical fashion [5]. ADIFOR transforms Fortran 77 programs using this approach.

To illustrate automatic differentiation with current ADIFOR, we differentiate the subroutine \texttt{torfcn} for the torsion problem that maps an \( n \)-vector \( x \) into a scalar \( f \). The vector \( x \) contains the independent variables, and the scalar \( f \) contains the dependent variable. The full code for \texttt{torfcn} can be found in the appendix.

The first loop (LOOP1) is shown in Figure 1. It computes \( S_1 \), whose value is stored in \texttt{fquad}. Currently, ADIFOR generates the code shown in Figure 2 for computing \( \frac{\partial f_{\text{quad}}}{\partial x} \). In accordance
```fortran
fquad = 0.0
do 20 j = 0, ny
    do 10 i = 0, nx
        k = nx*(j-1) + i
        vr = 0.0
        vu = 0.0
        if (i .ge. 1 .and. j .ge. 1) v = x(k)
        if (i .lt. nx .and. j .gt. 0) vr = x(k+1)
        if (i .gt. 0 .and. j .lt. ny) vu = x(k+nx)
        fquad = fquad + hyx*(vr-v)**2 + hxy*(vu-v)**2
10  continue
20  continue
```

Figure 1: Code for LOOP1

with the specification of ADIFOR (see [3]), g$p denotes the actual length of the derivative objects in a call to derivative code. Since Fortran 77 does not allow dynamic memory allocation, derivative objects for local variables are statically allocated with leading dimension pmax. pmax is specified by the user when ADIFOR processes the Fortran code for torfcn. A variable and its associated objects are treated in analogous manner; that is, if x is function parameter, so is g$x. Derivative objects corresponding to locally declared variables or variables in common blocks are declared as local variables or variables in common blocks. Given x and g$x, the derivative code computes

\[ g\#\text{fquad}(l: g\#p) = \left( \frac{df\text{quad}}{dx} \right) g\#\text{x}(l: g\#p, l: n)^T \]

In particular, if g$p equals n and g$x is the n \times n identity matrix, it computes the gradient of fquad with respect to x.

An active variable is one that is on the computational path from independent to dependent variables (see [4]). Notice that in the ADIFOR-generated code, a loop of length g$p is associated with every assignment statement involving an active variable. Therefore the cost of floating-point operations can be approximated as (g$p \times function evaluation). The storage requirement for ADIFOR-generated code is (g$p \times number of active variables). We note two key points about the current ADIFOR:

- ADIFOR uses the forward mode overall to compute derivatives. That is, ADIFOR code maintains the derivatives of intermediate variables with respect to all input variables. So, for example, g$v$u = \frac{dv}{dx}.
- ADIFOR uses the reverse mode to preaccumulate “local” derivatives for assignments statements.

The reverse mode is best understood with an example. For example, in the torsion problem, we have the assignment

\[ f\text{quad} = hyx * (vr - v) * *2 + hxy * (vu - v) * *2, \]
Figure 2: ADIFOR-generated Derivative Code for LOOP1
where \( h_{xy} \) and \( h_{yx} \) are constants. The chain rule tells us that
\[
\nabla f_{\text{quad}} = \frac{\partial f_{\text{quad}}}{\partial v_r} \ast \nabla v_r + \frac{\partial f_{\text{quad}}}{\partial v_u} \ast \nabla v_u + \frac{\partial f_{\text{quad}}}{\partial v} \ast \nabla v.
\]

Hence, if we know the “local” derivatives \( \left( \frac{\partial f_{\text{quad}}}{\partial v_r}, \frac{\partial f_{\text{quad}}}{\partial v_u}, \frac{\partial f_{\text{quad}}}{\partial v} \right) \) of \( f_{\text{quad}} \) with respect to \( v, v_u, \) and \( v_r \), we can easily compute \( \nabla w \), the derivatives of \( w \) with respect to \( x \). The “local” derivatives \( \left( \frac{\partial f_{\text{quad}}}{\partial v_r}, \frac{\partial f_{\text{quad}}}{\partial v_u}, \frac{\partial f_{\text{quad}}}{\partial v} \right) \) can be computed efficiently by using the reverse mode of automatic differentiation. Here we maintain the derivative of the final result with respect to an intermediate quantity. These quantities are usually called adjoints. They measure the sensitivity of the final result with respect to some intermediate quantity. In the reverse mode, let \( t_{\bar{t}} \) denote the adjoint object corresponding to \( t \). The goal is for \( t_{\bar{t}} \) to contain the derivative \( \frac{\partial f_{\text{quad}}}{\partial t} \). We know that \( w_{\bar{t}} = \frac{\partial f_{\text{quad}}}{\partial f_{\text{quad}}} = 1.0 \). We can compute \( v_{\bar{t}} \) and \( z_{\bar{t}} \) by applying the following simple rule to the statements executed in computing \( f_{\text{quad}} \), but in reverse order:

\[
\begin{align*}
\text{if } s = f(t), \text{ then } t_{\bar{t}} &= s_{\bar{t}} \ast (df/dt) \\
\text{if } s = f(t,u), \text{ then } t_{\bar{t}} &= s_{\bar{t}} \ast (df/dt) \\
&\hspace{1cm} u_{\bar{t}} &= s_{\bar{t}} \ast (df/du)
\end{align*}
\]

Using this simple recipe (see [14,20]), we generate the code shown in Figure 3 for computing \( v_{\bar{t}}, v_{r\bar{t}}, \) and \( v_{\bar{t}} \). One can easily convince oneself that
\[
\begin{align*}
v_{\bar{t}} &= 2 \ast h_{xy} \ast (v_r - v) \\
v_{r\bar{t}} &= 2 \ast h_{yx} \ast (v_r - v) \\
v_{\bar{t}} &= -2 \ast h_{xy} \ast (v_u - v) - 2 \ast h_{yx} \ast (v_r - v)
\end{align*}
\]

so that we have in fact computed the correct “local” derivatives. The code shown in Figure 2 has been generated by applying this same technique to all other assignments statements involving active variables and by optimizing the resulting code by removing additions with 0 and multiplications with 1. The ADIFOR-generated code for the whole subroutine is shown in Appendix B. ADIFOR is currently not consistent about pulling loop invariant subexpressions out of the loop, but will do so reliably in the future.

3 Extending the Scope of the Compile-Time Reverse Mode

In this section, we explore extensions of the compile-time reverse mode to

- a sequence of assignment statements,
- a nested loop, and
- the whole program.

A closer look at the current ADIFOR-generated code in the preceding sections reveals a substantial time and space overhead associated with the computations of auxiliary gradients such as \( g_{\$v}, g_{\$v_u}, \) and \( g_{\$v_r} \). In this section, we explore different ways for improving the overall computation of the gradient by extending the scope of the reverse mode.
/* Compute function values */
\[ d/0 = (v_r - v) \]
\[ d/4 = (v_u - v) \]
\[ t1 = d/0 \cdot d/0 \]
\[ t2 = t1 \cdot hxy \]
\[ t3 = d/4 \cdot d/4 \]
\[ t5 = t3 \cdot hxy \]
\[ w = t2 + t5 \]

/* Initialize adjoint quantities */
\[ \bar{w} = 1.0; \bar{t3} = 0.0; \bar{t2} = 0.0; \]
\[ \bar{t1} = 0.0; \bar{d/0} = 0.0; \bar{d/4} = 0.0; \]

/* Adjoints for \( w = t2 + t5 \) */
\[ \bar{t2} = \bar{t2} + \bar{w} \]
\[ \bar{t5} = \bar{t5} + \bar{w} \]

/* Adjoints for \( t5 = t3 \cdot hxy \) */
\[ \bar{t3} = \bar{t3} + \bar{t5} \cdot hxy \]

/* Adjoints for \( t3 = d/4 \cdot d/4 \) */
\[ d/4 = d/4 + \bar{t3} \cdot d/4 \]
\[ d/4 = d/4 + \bar{t3} \cdot d/4 \]

/* Adjoints for \( t2 = t1 \cdot hxy \) */
\[ \bar{t1} = \bar{t1} + \bar{t2} \cdot hxy \]

/* Adjoints for \( d/0 = (v_r - v) \) */
\[ \bar{v_r} = \bar{v_r} + \bar{d/0} \cdot 1 \]

Figure 3: Unoptimized Reverse Mode Computation
3.1 Case 1: Reverse Mode for Basic Blocks inside the Loop

In the program for the torsion problem, there are three loops: two for the computation of the quadratic part of the function and one for the computation of the linear part. Consider, for example, LOOP1. Each loop iteration can be viewed as a mapping

\[ [x(k), x(k+1), x(k+n), f\text{quad}_\text{old}] \longrightarrow f\text{quad}_\text{new}. \]

We use the notation \( f\text{quad}_\text{old} \) and \( f\text{quad}_\text{new} \) to distinguish between the original and updated value of the variable \( f\text{quad} \). Hence, if we know

\[
\frac{\partial f\text{quad}_\text{new}}{\partial x(k)} \frac{\partial f\text{quad}_\text{new}}{\partial x(k+1)} \frac{\partial f\text{quad}_\text{new}}{\partial x(k+nx)} \text{ and } \frac{\partial f\text{quad}_\text{new}}{\partial f\text{quad}_\text{old}}
\]

then we can update \( \nabla f\text{quad} \) as follows:

\[
\nabla f\text{quad} = \frac{\partial f\text{quad}_\text{new}}{\partial x(k)} \nabla f\text{quad} + \frac{\partial f\text{quad}_\text{new}}{\partial x(k+1)} \nabla x(k+1) \\
+ \frac{\partial f\text{quad}_\text{new}}{\partial x(k+nx)} \nabla x(k+nx) + \frac{\partial f\text{quad}_\text{new}}{\partial f\text{quad}_\text{old}} \nabla f\text{quad}.
\]

The derivatives in the equation (2) can easily be computed by applying the reverse mode to the loop body. The resulting code is shown in Figure 4. Note that each variable is assigned only once in each loop iteration. If this had not been the case, we would have had to save the sequence of values of variables that are overwritten by allocating some extra temporary variables. This extension of the scope of the reverse mode saved us 3 derivative vectors \( g_{\text{sv}}, g_{\text{svr}}, \) and \( g_{\text{svu}} \), and decreased the number of derivative vector operations from 10 to 4.

In general, we can apply this technique in a straightforward fashion to any piece of code that has only one entry and exit point and does not contain subroutine or function calls or loops. We call such a piece of code a basic block. We may have to introduce some temporaries to make sure that each variable is assigned only once (i.e., represents a unique value) in a basic block, but this requires at most as many scalar temporaries as there are lines of code, an insignificant increase of storage. The savings achieved by this technique depend on the particular code at hand, but, in general, will be the more pronounced the more statements a basic block contains. The code that results from applying this technique to the whole subroutine is shown in Appendix C.

3.2 Case 2: Reverse Mode for the Whole Loop

In order to expand the scope of the compile-time reverse mode, the special structure of the torsion problem is important. Defining

\[ t_k := -hxy \ast (vx - v) \ast \ast 2 + hxy \ast (vu - v) \ast \ast 2 \]

to be the value computed in loop iteration \( k \) to upgrade \( f\text{quad} \), we can express

\[
f\text{quad} = \sum_{k=1}^{(nx+1)(ny+1)} t_k. \tag{3}
\]
FQUAD = 0.0
DO G$1$ = 1,G$P$
G$9FQUAD(G$1$) = 0.0d0
END DO
DO 99998 J = 0,NY
   DO 99999 I = 0,NX
      K = NX*(J-1)+I
      c compute new contribution to sum
      V = 0.0
      VR = 0.0
      VU = 0.0
      IF (I.GE.1.AND. J.GE.1) THEN
         V = X(K)
      END IF
      IF (I.LT.NX.AND. J.GT.0) THEN
         VR = X(K+1)
      END IF
      IF (I.GT.0.AND. J.LT.NY) THEN
         VU = X(K+NX)
      END IF
      c reverse mode computation for computing derivatives of
      c x(k), x(k+1), x(k+nx). We know that the deriv. of fquad_new
      c with respect to fquad_old is 1.
      c
      D$0$ = VR - V
      D$4$ = VU - V
      FQUAD = FQUAD + HYX*D$0**2 + HXY*D$4**2
      VBAR = -2*MHYX*D$4 - 2*MHYX*D$0
      VBAR = 2*MHYX*D$4
      VRBAR = 2*MHYX*D$0
      XKBAR = 0.0
      XK1BAR = 0.0
      XKNXBAR = 0.0
      IF (I.GE.1.AND. J.GE.1) XKBAR = VBAR
      IF (I.LT.NX.AND. J.GT.0) XK1BAR = VRBAR
      IF (I.GT.0.AND. J.LT.NY) XKNXBAR = VUBAR
      c Chain Rule to update derivatives of fquad w.r.t. x
      c
      DO PP = 1,G$P$
         G9FQUAD(PP) = G9FQUAD(PP) + XKBAR*G$X(PP,K) +
         + XK1BAR*G$X(PP,K+1) +
         + XKNXBAR*G$X(PP,K+NX)
      END DO
99999 CONTINUE
20 CONTINUE
99998 CONTINUE

Figure 4: Reverse Mode for Basic Block in LOOP1
Since $v$, $vu$, and $v\tau$ are defined in terms of $x(k+1)$, $x(k+nx)$, and $x(k)$, $t_k$ is a function of these values, that is,

$$t_k = t_k(x(k+1), x(k+nx), x(k)).$$

Since no entry of $x$ is overwritten in any of the loop iterations, $t_k$ and $t_l$ do not depend on each other for $k \neq l$, and we can compute the sum (3) in any order. In compiler terms, there are no loop-carried dependencies and this loop is a parallel loop.

Remember that the reverse mode implicitly assumes that we are able to trace the values computed during some computation in the reverse order. Hence, a loop that is not parallel would require us to save some intermediate values. However, for a parallel loop, it is sufficient simply to generate the reverse mode code for the loop body. But this is exactly what we already did in the preceding section, where we computed

$$\frac{\partial t_k}{\partial x(k+1)} \frac{\partial t_k}{\partial x(k)} \frac{\partial t_k}{\partial x(k+nx)}.$$ 

Now, since $t_l$ and $t_k$ do not depend on each other for $l \neq k$, the associativity of addition allows us to compute

$$\frac{df_{\text{quad}}}{dx(j)} = \frac{\partial t_{j-1}}{\partial x(j)} + \frac{\partial t_{j-nx}}{\partial x(j)} + \frac{\partial t_j}{\partial x(j)}$$

in a piecemeal fashion, as each of the iterations $j$, $j-1$, and $j-nx$ contributes to $\frac{df_{\text{quad}}}{dx(j)}$. The resulting code is shown in Figure 5. The $xbar$ vector contains $\frac{df_{\text{quad}}}{dx}$ and components $k+1$, $k$, and $k+nx$ are updated in iteration $k$. After the loop, we apply the chain rule to compute

$$\nabla f_{\text{quad}} = \frac{df_{\text{quad}}}{dx} \cdot \nabla x.$$ 

This matrix-vector multiplication is performed using the BLAS routine DGEMV [13].

To summarize, we exploited the fact that

- loop iterations do not depend on each other, and
- the result of each loop enters into the dependent variable ($f_{\text{quad}}$) in an additive fashion.

This allowed us to generate reverse mode code for the whole loop by simply generating reverse mode code for the loop body, and the forward mode propagation of the global derivatives could be moved outside of the loop.

Compared with the code in the previous section, we now have a multiplication of an $g \times (nx+1)$ ($ny+1$) matrix by a vector outside the loop instead of $(nx+1)(ny+1)$ multiplications of an $g \times (nx+1)$ matrix by a vector multiplication inside a loop that is executed $(nx+1)$ ($ny+1$) times, requiring roughly one-fourth the number of operations. Applying this technique to the whole subroutine results in the code shown in Appendix D.

3.3 Case 3: The Full Reverse Mode

So far we exploited only the particular structure of the code in LOOP1, LOOP2, and LOOP3. On the other hand, $f(x)$ is the sum of the contributions computed in LOOP1, LOOP2, LOOP3,
FQUAD = 0.0
DO G$i$/& = 1,GS$P$
   G$FQUAD(G$i$/) = 0.0d0
END DO
DO I = 1,XBARSIZE
   XBAR(I) = 0.0
END DO
DO 99998 J = 0,NY
   DO 99999 I = 0,NX
      K = NX*(J-1) + I
      V = 0.0
      VR = 0.0
      VU = 0.0
      IF (I.GE.1 .AND. J.GE.1) V = X(K)
      IF (I.LT.NX .AND. J.GT.0) VR = X(K+I)
      IF (I.GT.0 .AND. J.LT.NY) VU = X(K+NX)
      fquad = fquad + hyx*(vr - v)**2 + hxy*(vu - v)**2
      D$\theta$/0 = VR - V
      D$\phi$/4 = VU - V
      FQUAD = FQUAD + HYX*D$\theta$/0**2 + HXY*D$\phi$/4**2
      VBAR = -2*HXY*D$\phi$/0 - 2*HYX*D$\theta$/0
      VUBAR = 2*HXY*D$\phi$/4
      VRBAR = 2*HYX*D$\theta$/0
      IF (I.GE.1 .AND. J.GE.1) XBAR(K) = XBAR(K) + VBAR
      IF (I.LT.NX .AND. J.GT.0) XBAR(K+1) = XBAR(K+1) + VRBAR
      IF (I.GT.0 .AND. J.LT.NY) XBAR(K+NX) = XBAR(K+NX) + VUBAR
99999 CONTINUE
99998 CONTINUE
C  
C xbar is the vector of partial derivatives of the contribution C to fquad with respect to x. Since fquad was zero before this C loop, the derivative d$FQUAD/dx = g$\phi$/ * xbar.
C  
CALL DGEMV( 'n', GS$P$,XBARSIZE,1.0d0,G$X$,LDG$X$,XBAR,1,1.0d0,GS$FQUAD, +                 1)

Figure 5: Reverse Mode for the Whole Loop
and, in addition to being parallel loops themselves, these loops do not depend on each other. So, instead of computing

\[
xbar(1:n) = 0;
/* Update xbar in LOOP1 */
d$d_quad = g$x * xbar
\]

\[
xbar(1:n) = 0;
/* Update xbar in LOOP2 */
d$d_quad = d$d_quad + g$x * xbar
\]

we could simply keep on updating xbar in LOOP1, LOOP2, and LOOP3 and perform compute

\[d$f_quad = g$x \ast xbar\]

once at the end. This is possible since none of these loops updates the vector \(x\), and hence \(g$x\) remains unchanged. But we can go even further: Since in the forward mode, \(g$x\) is initialized to the identity, we can eliminate the final multiplication \(g$x_quad = g$x \ast xbar\) and simply assign return xbar. In this fashion, we have generated adjoint code for the whole subroutine, and the code for computing the gradient does not contain any vector operations.

It is important to note that we were able to do the full implementation of the reverse mode because

- each of the three loops is a parallel loop,
- the three loops do not depend on each other,
- the contribution computed inside each loop enters the final result in an additive fashion, and
- the results of each of the three loops are added to achieve the final result.

The resulting reverse mode code for the torsion problem is shown in Appendix E. While we did not decrease the storage requirement any further compared with the preceding section, we saved another three loops of size \(g$p\) \(nx ny\), and the run time of this program no longer depends on \(g$p\).

4 Exploring Partial Separability

As was mentioned in the introduction, the torsion problem is a partially separable function \(f : \mathbb{R}^n \rightarrow \mathbb{R}\), in that it can be expressed as

\[f(x) = \sum_{i=1}^{n_b} f_i(x).\]

This structure can also be used advantageously in computing the (usually dense) gradient \(\nabla f\) of \(f\) (see [9]). Assume that the code for computation of \(f\) looks as follows:
subroutine f(n,x,fval)
  integer n
  real x(n), fval, temp
  fval = 0
  call ff(n,x,temp)
  fval = fval + temp
  
  ......
  call fnb(n,x,temp)
  fval = fval + temp
  return
end

If we submit f to ADIFOR, it generates

subroutine g$fn(n,x,g$x,ldg$x,fval,lg$fval,ldg$fval).

To compute $\nabla f$, the first (and only) row of the Jacobian of $f$, we set $g$x = n and initialize $g$x to a $n \times n$ identity matrix. Hence, in current ADIFOR, the cost of computing $\nabla f$ is of the order of $n$ times the function evaluation.

As an alternative, we realize that with $f : \mathbb{R}^n \rightarrow \mathbb{R}^{n_b}$ defined as

$$g = \begin{pmatrix} f_1 \\ \vdots \\ f_{n_b} \end{pmatrix},$$

we have the identities

$$f(x) = \epsilon^T g(x),$$

and hence $\nabla f(x) = \epsilon^T J_g$, where $\epsilon$ is the vector of all ones, and $J_g$ is the Jacobian of $g$. However, if most of the component functions $f_i$ depend only on a few parameters $x_j$, the Jacobian of $g$ is sparse, and this fact can be exploited advantageously.

The idea is best understood with an example. Assume that we have a function

$$F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{pmatrix} : x \in \mathbb{R}^4 \mapsto y \in \mathbb{R}^5$$

whose Jacobian $J$ has the following structure (symbols denote nonzeros, and zeros are not shown):

$$J = \begin{pmatrix} \bigdiamond & \bigdiamond \\ \bigtriangleup & \bigcirc & \bigcirc \\ \bigtriangleup & \bigcirc & \bigtriangleup & \bigcirc \\ \bigtriangleup & \bigcirc & \bigtriangleup & \bigcirc \\ \bigtriangleup & \bigcirc & \bigtriangleup & \bigcirc \end{pmatrix}.$$
That is, the function \( f_1 \) depends only on \( x_1 \), \( f_2 \) depends only on \( x_1 \) and \( x_4 \), and so on. The key idea in computing sparse Jacobians is to identify so-called structurally orthogonal columns \( j_i \) of \( J \) (see [10]), that is, columns whose inner product is always zero, independent of the numerical values of their nonzero entries. In our example, columns 1 and 2 are structurally orthogonal, and so are columns 3 and 4. This means that the set of functions that depend nontrivially on \( x_1 \), namely \( \{f_1, f_2\} \), and the set of functions that depend nontrivially on \( x_2 \), namely \( \{f_3, f_4, f_5\} \), are disjoint. Because of the graph-coloring approaches that are used to reveal this structure, one usually associates a “color” with every set of structurally orthogonal columns.

To exploit this sparsity structure, we recall that ADIFOR (ignoring transposes) computes \( J \cdot S \), where \( S \) is a matrix with \( g \times p \) columns. For our example, setting \( S = I_{4 \times 4} \) will give us \( J \) at roughly four times the cost of evaluating \( f \), but if we exploit the structural orthogonality and set

\[
S = \begin{pmatrix}
1 & 0 \\
1 & 0 \\
0 & 1 \\
0 & 1
\end{pmatrix},
\]

the running time for the ADIFOR code is roughly halved. The ADIFOR-generated code remains unchanged.

This idea can readily be applied to the torsion problem. By storing the contribution of iteration \( k \) to \( f_{\text{quad}} \) in the \( k \)-th element of separate vectors \( FQ, FQQ, \) and \( FP \) (for the LOOP1, LOOP2, and LOOP3, respectively), the derivative of \( f_{\text{quad}} \) is the sum of the row sums of the Jacobians of \( FQ, FQQ, \) and \( FP \).

For example, the code for the loop corresponding to \( FQ \) is

```fortran
SUBROUTINE TORFCN1(N,X,X,F,NX,NY,HX,HY,FORCE,FQ)
  c       .. Scalar Arguments ..
  DOUBLE PRECISION F,FORCE,HX,HY
  INTEGER N,NX,NY
  c       .. Array Arguments ..
  DOUBLE PRECISION FQ(*),X(N)
  c       .. Local Scalars ..
  DOUBLE PRECISION FQUAD,HXY,HYX,V,VR,VU
  INTEGER FQK,I,J,K
  c
  HXY = HX/HY
  HYX = HY/HX
  FQUAD = 0.0
  DO 20 J = 0,NY
  DO 10 I = 0,NX
    K = NX* (J-1) + I
    V = 0.0
    VR = 0.0
```

The only change (compared with the corresponding code fragment in Appendix A) is that we replaced the accumulation of $f_{quad}$ by an assignment to $FQ$. Subroutines torfcn2 and torfcn3 to compute $FQQ$ and $FP$, respectively, are generated in the same fashion. For these codes, ADIFOR then generates the derivative codes shown in Appendix F.

For $n = 40$, the structures of $\frac{\partial FQ}{\partial x}$ and $\frac{\partial FQQ}{\partial x}$ are shown in Figure 6, and $\frac{\partial FP}{\partial x}$ is diagonal. The Jacobian for $FQ$ and $FQQ$ can be grouped into three sets of structurally orthogonal columns, independent of the size of the problem. And in the case of the function $FP$, the Jacobian can be compressed into only one column.

Exploiting this structure, we can now initialize the gradient vector as follows:
CALL SPARSITY(N,X,F,NX,NY,HX,HY,INDROWQ,INDCOLQ,NNZQ,INDROWQQ,
+    INDCOLQQ,NNZQQ)

DO I = 1,NNZQ
    INDRWQS(I) = INDROWQ(I)
    INDCOLQS(I) = INDCOLQ(I)
    INDRWQQS(I) = INDROWQQ(I)
    INDCOLQQS(I) = INDCOLQQ(I)
END DO

CALL DSM((NY+1)*(NX+1),N,NNZQ,INDROWQ,INDCOLQ,NGRPQ,MGRPQ,+
+    MINGRPQ,INFO,IPNTRQ,JPNTRQ,IWA,LIWA)

CALL DSM((NX+1)*(NY+1),N,NNZQQ,INDROWQQ,INDCOLQQ,NGRPQQ,+
+    MAXGRPQQ,MINGRPQQ,INFO,IPNTRQQ,JPNTRQQ,IWA,LIWA)

* ********************
*  * compute Jacobians for the individual loops *
*  ********************

c----- calc g$FQ
    DO I = 1,N
        DO J = 1,MAGRPFQ
            G$X(J,I) = 0
        END DO
        G$X(NGRPQ(I),I) = 1.0
    END DO
    CALL REV0A(MAXGRPQ,N,X,G$X,PX,FX,NX,NY,HX,HY,FORCE,FQ,+
        G$FQ,MAXCOLOR)

c------ calc g$FQQ
    DO I = 1,N
        DO J = 1,MAGRPFQQ
            G$X(J,I) = 0
        END DO
        G$X(NGRPQQQ(I),I) = 1.0
    END DO
    CALL REV0B(MAXGRPQQQ,N,X,G$X,PX,FX,NX,NY,HX,HY,FORCE,FQQ,+
        G$FQQ,MAXCOLOR)

c------ calc G$FP
After we have initialized some arrays determining the sparsity pattern of the Jacobian, we call the MINPACK subroutine DSM [9] to determine the proper coloring for the Jacobians of $F_Q$ and $F_{QQ}$. Having thus determined $NGRPQ(i)$, the “color” of column $i$ and $MAXGRPQ$, the number of colors for the Jacobian of $F_Q$, we initialize $g$ and calls rev0a (a renamed version of the ADIFOR-generated subroutine for torfcn1) to compute the compressed Jacobian of $F_Q$. The same idea is applied to compute $g_{FQQ}$ and $g_{FP}$. Lastly, the derivative values of the subfunctions are all added into a sparse vector, without ever expanding the compressed component Jacobians, as shown below. For the Jacobian of $F_Q$, the index arrays INDRGWQS and INDCOLQS indicate the row and column index of nonzero entries, and the NGRPQ array indicates the group (corresponding to one particular color) of a certain column. The Jacobian of $F_{QQ}$ is dealt with accordingly. The uncompression of the Jacobian...
of $F_0$ is trivial, since it was diagonal — we just add the $i$-th diagonal entry (properly scaled) to the $i$-th entry of the gradient accumulation vector $\text{SPARSEGF}$. The MINPACK documentation contains details on the particular data structures used to represent the sparse derivative matrices.

We note that we could of course apply the idea of the "basic block reverse mode" to generate improved derivative code for $\text{torfcn1}$, etc. This code is shown in Appendix G. We would expect much less spectacular savings in this case, since the length of the derivative objects was not more than three for our sparse Jacobians (whereas it was $n$ when we did not exploit partial separability).

5 Experimental Results

We tested the performance of our various derivative codes on a Sun Sparcstation iPX with 48 Mbytes of memory and an IBM RS6000/550 with 128 Mbytes of memory. We computed gradients for $n = 10 \times 10, 20 \times 20, \ldots, 100 \times 100$. For the alternatives described in Sections 2, 3.1, and 3.2, we computed gradients in slices of 10 elements (i.e., the gradient was computed by calling the derivative code $\lceil n/10 \rceil$ times). Figure 7 shows the ratio of the run time of a gradient to a function evaluation obtained for these derivative codes. As expected, the run time is linear in $n$, but the slope decreases as we expand the scope of the reverse mode.

In Figure 8 we show the ratio of the run time of a gradient to a function evaluation obtained by the full reverse mode (Section 3.3) and by exploiting the partial separability of the torsion problem. These graphs also show the run time of the hand-coded derivative subroutine supplied in the MINPACK-2 test suite. We see that by exploiting partial separability, we can achieve very good performance for computing the gradient of the torsion problem. This is particularly noteworthy as we do not need to know anything more about the structure of the problem than that it is partially separable. In contrast, intimate knowledge of the code is required to develop the full reverse mode and the hand-coded versions.

Acknowledgments

We thank Alan Carle, George Corliss, and Andreas Griewank for their comments. We are especially grateful to George Corliss for his thoughtful and thorough comments on an earlier draft of this paper.

References


Figure 7: Performance of Current ADIFOR Code and Enhanced Versions
Figure 8: Exploiting Partial Separability, the Full Reverse Mode and Handcoded Derivatives


A  Minpack-2 Code for the Torsion Problem

SUBROUTINE TORFCN(N,F,HX,NX,HY,NY,FORCE)
  c This subroutine computes the function of the torsion problem.
  c The spacing parameters hx and hy are for a rectangle with
  c nx points on the x-axis and ny points on the y-axis
  c .. Scalar Arguments ..
  DOUBLE PRECISION F,FORCE,HX,HY
  INTEGER N,NX,NY
  c ..
  c .. Array Arguments ..
  DOUBLE PRECISION X(N)
  c ..
  c .. Local Scalars ..
  DOUBLE PRECISION FLIN,FQUAD,HXY,V,VR,VU
  INTEGER I,J,K
  c
  HXY = HX/HY
  HYX = HY/HX
  c
  Computation of the quadratic part of the function.
  c
  LOOP1:

  FQUAD = 0.0
  DO 20 J = 0,NY
     DO 10 I = 0,NX
          K = NX* (J-1) + I
          V = 0.0
          VR = 0.0
          VU = 0.0
          IF (I. GE. 1  .AND. J. GE. 1) V = X(K)
          IF (I. LT. NX  .AND. J. GT. 0) VR = X(K+1)
          IF (I. GT. 0  .AND. J. LT. NY) VU = X(K+NX)
          FQUAD = FQUAD + HXY* (VR-V)**2 + HYX* (VU-V)**2
  10    CONTINUE
  20    CONTINUE
  c
  LOOP2:
  c
  DO 40 J = 1,NY + 1
DO 30 I = 1,NX + 1
   K = NX* (J-1) + I
   V = 0.0
   VL = 0.0
   VD = 0.0
   IF (I.LE.NX .AND. J.LE.NY) V = X(K)
   IF (I.GT.1 .AND. J.LE.NY) VL = X(K-1)
   IF (I.LE.NX .AND. J.GT.1) VD = X(K-NX)
   FQUAD = FQUAD + HYX* (VL-V)**2 + HXY* (VD-V)**2
30 CONTINUE
CONTINUE

c Computation of the linear part of the function.
c
LOOP 3:

FLIN = 0.0
DO 50 K = 1,NX*NY
   FLIN = FLIN + X(K)
50 CONTINUE
F = 0.25*FQUAD - FORCE*HX*HY*FLIN

END
SUBROUTINE REV0(G$PS,N,X,G$SX,LDG$X,F,G$F,LDG$F,NX,NY,HX,HY,FORCE)

*******************************************************************************
* *
generated by current ADIFOR for computing gradient of
* torsion problem. x independent, f dependent.
*
*******************************************************************************

C ADIFOR: runtime gradient index
C ADIFOR: translation time gradient index
C ADIFOR: gradient iteration index
C

*******
C This subroutine computes the function of the torsion problem.
*******
C The spacing parameters hx and hy are for a rectangle with
C nx points on the x-axis and ny points on the y-axis
C
C ADIFOR: gradient declarations
C .. Parameters..
INTEGER G$PMAX
PARAMETER (G$PMAX=4900)
C ..
C .. Scalar Arguments..
DOUBLE PRECISION F,FORCE,HX,HY
INTEGER G$PS,LDG$F,LDG$X,N,NX,NY
C..
C .. Array Arguments..
DOUBLE PRECISION G$F(LDG$F),G$X(LDG$X,N),X(N)
C ..
C .. Local Scalars..
DOUBLE PRECISION D$0,D$4,FLIN,FQUAD,HXY,HYX,V,WD,VL,VR,VU
INTEGER G$S$18,I,J,K
C ..
C .. Local Arrays..
DOUBLE PRECISION G$FLIN(G$S$PMAX),G$FQUAD(G$S$PMAX),G$V(G$S$PMAX),
+ G$VD(G$S$PMAX),G$VL(G$S$PMAX),G$VR(G$S$PMAX),
+ G$VU(G$S$PMAX)
C..
IF (G$P$.GT.G$PMAX) THEN
   PRINT *, 'Parameter ge is greater than gpmax.'
   STOP
END IF
HXY = HX/HY
HYX = HY/HX
C Computation of the quadratic part of the function.
FQUAD = 0.0
DO G$1$ = 1,G$P$
   G$FQUAD$(G$1$) = 0.0d0
END DO
DO 99998 J = 0,NY
   DO 99999 I = 0,NX
      K = NX* (J-1) + I
      V = 0.0
      DO G$1$ = 1,G$P$
         G$V$(G$1$) = 0.0d0
      END DO
      VR = 0.0
      DO G$1$ = 1,G$P$
         G$VR$(G$1$) = 0.0d0
      END DO
      VU = 0.0
      DO G$1$ = 1,G$P$
         G$VU$(G$1$) = 0.0d0
      END DO
      IF (I.GE.1 .AND. J.GE.1) THEN
         V = x(k)
         DO G$1$ = 1,G$P$
            G$V$(G$1$) = G$X$(G$1$ , K)
         END DO
         V = X(K)
      END IF
      IF (I.LT.NX .AND. J.GT.0) THEN
         VR = x(k + 1)
         DO G$1$ = 1,G$P$
            G$VR$(G$1$) = G$X$(G$1$ , K+1)
         END DO
         VR = X(K+1)
      END IF
      IF (I.GT.0 .AND. J.LT.NY) THEN
         VU = x(k + nx)
         DO G$1$ = 1,G$P$
            G$VU$(G$1$) = G$X$(G$1$ , K+nx)
         END DO
         VU = X(K+nx)
      END IF
      C
      fquad = fquad + hyx * (vr - v) ** 2 + hxy * (vu - v) ** 2
      D$0$ = VR - V
      D$4$ = VU - V
      DO G$1$ = 1,G$P$
         G$FQUAD(G$1$) = G$FQUAD(G$1$) + hyx*2*D$0$*G$VR(G$1$) +
                            (- (hxy*2*D$4$)- (hxy*2*D$0$)) *
                            G$V(G$1$) + hxy*2*D$4$*G$VU(G$1$)
FQUAD = FQUAD + HYX*D$0**2 + HXY*D$4**2

10  CONTINUE

99999  CONTINUE

20  CONTINUE

DO 9996 J = 1,NY + 1
    DO 9997 I = 1,NX + 1
        K = NX* (J-1) + I
        V = 0.0
        DO G$1$ = 1,G$P$
            G$V(G$1$) = 0.0d0
        END DO
        VL = 0.0
        DO G$1$ = 1,G$P$
            G$VL(G$1$) = 0.0d0
        END DO
        VD = 0.0
        DO G$1$ = 1,G$P$
            G$VD(G$1$) = 0.0d0
        END DO
        IF (I.LE.NX .AND. J.LE.NY) THEN
            V = x(K)
            DO G$1$ = 1,G$P$
                G$V(G$1$) = G$X(G$1$ , K)
            END DO
            V = x(K)
        END IF
        IF (I.GT.1 .AND. J.LE.NY) THEN
            VL = x(K-1)
            DO G$1$ = 1,G$P$
                G$VL(G$1$) = G$X(G$1$ , K-1)
            END DO
            VL = x(K-1)
        END IF
        IF (I.LE.NX .AND. J.GT.1) THEN
            VD = x(K-NX)
            DO G$1$ = 1,G$P$
                G$VD(G$1$) = G$X(G$1$ , K-NX)
            END DO
            VD = x(K-NX)
        END IF
        IF (fquad = fquad + hyx * (vl - v) ** 2 + hxy * (vd - v) ** 2
            D$0 = VL - V
            D$4 = VD - V
            DO G$1$ = 1,G$P$
                G$FQUAD(G$1$) = G$FQUAD(G$1$) + HYX*2*D$0*G$VL(G$1$) + G$FQUAD(G$1$) = G$FQUAD(G$1$) + HYX*2*D$0*G$VL(G$1$) +
+ (- (HXY*2*D$4) - (HXY*2*D$0)) * 
+ G$V(G$1$) + HXY*2*D$4*G$VD(G$1$)

END DO

FQUAD = FQUAD + HXY*D$0**2 + HXY*D$4**2

30 CONTINUE

99997 CONTINUE

40 CONTINUE

99996 CONTINUE

C Computation of the linear part of the function.

FLIN = 0.0

DO G$1$ = 1,G$P$
   G$FLIN(G$1$) = 0.0d0
END DO

DO 99995 K = 1,NX*NY

C flin = flin + x(k)

DO G$1$ = 1,G$P$
   G$FLIN(G$1$) = G$FLIN(G$1$) + G$X(G$1$,K)
END DO

FLIN = FLIN + X(K)

50 CONTINUE

99995 CONTINUE

C f = 0.25 * fquad - force * hx * hy * flin

DO G$1$ = 1,G$P$
   G$F(G$1$) = 0.25*G$FQUAD(G$1$) - (FORCE*HX*HY)*G$FLIN(G$1$)
END DO

F = 0.25*FQUAD - FORCE*HX*HY*FLIN

END
C Reverse Mode for Basic Blocks

SUBROUTINE REV1(G$P$,N,X,G$X$,LDG$X$,F,G$F$,LDG$F$,NX,NY,HX,HY,FORCE)

**********************************************************************
* * reverse mode at basic block level
* **********************************************************************

C ADIFOR: runtime gradient index
C ADIFOR: translation time gradient index
C ADIFOR: gradient iteration index

********
This subroutine computes the function of the torsion problem.

********
The spacing parameters hx and hy are for a rectangle with
nx points on the x-axis and ny points on the y-axis

C ADIFOR: gradient declarations
.
.. Parameters..
INTEGER G$PMAX$
PARAMETER (G$PMAX$=10000)
.
.. Scalar Arguments..
DOUBLE PRECISION F,FORCE,HX,HY
INTEGER G$P$,LDG$F$,LDG$X$,N,NX,NY
.
.. Array Arguments..
DOUBLE PRECISION G$F$(LDG$F$),G$X$(LDG$X$,N,NX,NY)
.
.. Local Scalars..
DOUBLE PRECISION D$0$,D$4$,FLIN,FLINBAR,FQUAD,HXY,HYX,V,VBAR,VD,
+ VDBAR,VL,VLEBAR,VR,VRBAR,VU,VUBAR,XK,BAR,XKBAR,
+ XKNXBAR
INTEGER G$IS$,I,J,K,PP
.
.. Local Arrays..
DOUBLE PRECISION G$FLIN$(G$PMAX$),G$FQUAD$(G$PMAX$)
.
IF (G$P$.GT.G$PMAX$) THEN
   PRINT *, 'Parameter g$P is greater than g$PMAX.'
   STOP
END IF
HXY = HX/HY
HYX = HY/HX

Computation of the quadratic part of the function.
FQUAD = 0.0
DO GS1S = 1,GSFS
   GSFQUAD(GS1S) = 0.0d0
END DO
DO 99998 J = 0,NY
   DO 99999 I = 0,NX
      K = NX* (J-1) + I
      V = 0.0
      VR = 0.0
      VU = 0.0
      IF (I.GE.1 .AND. J.GE.1) THEN
         V = X(K)
      END IF
      IF (I.LT.NX .AND. J.GT.0) THEN
         VR = X(K+1)
      END IF
      IF (I.GT.0 .AND. J.LT.NY) THEN
         VU = X(K+NX)
      END IF
      C fquad = fquad + hyx * (vr - v) ** 2 + hxy * (vu - v) ** 2
      FQUAD = FQUAD + HYX*D$0**2 + HXY*D$4**2
      VBAR = -2*HXY*D$4 - 2*HYX*D$0
      VUBAR = 2*HXY*D$4
      VRBAR = 2*HYX*D$0
      XXBAR = 0.0
      X1BAR = 0.0
      XXNBAR = 0.0
      IF (I.GE.1 .AND. J.GE.1) XXBAR = VBAR
      IF (I.LT.NX .AND. J.GT.0) X1BAR = VBAR
      IF (I.GT.0 .AND. J.LT.NY) XXNBAR = VBAR
C Chain Rule to update derivatives of fquad w.r.t. x
DO PP = 1,GSFS
   GSFQUAD(PP) = GSFQUAD(PP) + XXBAR*GSX(PP,K) +
END DO

CONTINUE

DO 99996 J = 1, NY + 1
  DO 99997 I = 1, NX + 1
    K = NX* (J-1) + I

    V = 0.0
    VL = 0.0
    VD = 0.0
    IF (I.LE.NX .AND. J.LE.NY) V = X(K)
    IF (I.GT.1 .AND. J.LE.NY) VL = X(K-1)
    IF (I.LE.NX .AND. J.GT.1) VD = X(K-NX)

    fquad = fquad + hyx * (vl - v) ** 2 + hxy * (vd - v) ** 2

    D$0 = VL - V
    D$4 = VD - V
    FQUAD = FQUAD + HYX*D$0**2 + HXY*D$4**2
    VBAR = -2*HX*D$4 - 2*HY*H$D
    VDBAR = 2*HX*H$D
    VLRBAR = 2*HY*H$D
    XKBAR = 0.0
    XK1BAR = 0.0
    XKNXBAR = 0.0
    IF (I.LE.NX .AND. J.LE.NY) XKBAR = VBAR
    IF (I.GT.1 .AND. J.LE.NY) XK1BAR = VLRBAR
    IF (I.LE.NX .AND. J.GT.1) XKNXBAR = VDBAR

  DO PP = 1, GSFS
    GS$FQUAD(PP) = GS$FQUAD(PP) + XKBAR*GSX(PP,K) +
      XK1BAR*GSX(PP,K-1) +
      XKNXBAR*GSX(PP,K-NX)
  END DO

CONTINUE

CONTINUE

C Computation of the linear part of the function.

FLIN = 0.0
DO GSFS = 1, GSFS
  GS$FLIN(GSFS) = 0.0d0
END DO

31
DO 99995 K = 1, NX*NY
   cm
      g$flin(k) = g$flin(k) + 1
      FLINBAR = 1.0
      DO I = 1, G$P$
         G$FLIN(I) = G$FLIN(I) + FLINBAR*G$X(I,K)
      END DO
      FLIN = FLIN + X(K)
END 99995 CONTINUE
C f = 0.25 * fquad - force * hx * hy * flin
DO 1 G$S$I$ = 1, G$S$P$
   G$F(G$S$I$) = 0.25*G$FQUAD(G$S$I$) - (FORCE*HX*HY)*G$FLIN(G$S$I$)
END DO
F = 0.25*FQUAD - FORCE*HX*HY*FLIN
END
D Reverse Mode for Loop Bodies

SUBROUTINE REV2(G$P$,N,X,G$X$,LDG$X$,F,G$F$,LDG$F$,NX,NY,HX,HY,FORCE,
+ XBAR,XBARSIZE)

*******************************
* reverse mode at individual loop level
*******************************

C ADIFOR: runtime gradient index
C ADIFOR: translation time gradient index
C ADIFOR: gradient iteration index

**********
This subroutine computes the function of the torsion problem.
**********
The spacing parameters hx and hy are for a rectangle with
nx points on the x-axis and ny points on the y-axis

C ADIFOR: gradient declarations

.. Parameters ..
INTEGER G$P$MAX$\$
PARAMETER (G$P$MAX$\$=10000)

.. Scalar Arguments ..
DOUBLE PRECISION F,FORCE,HX,HY
INTEGER G$P$,LDG$F$,LDG$X$,N,NX,NY,XBARSIZE

.. Array Arguments ..
DOUBLE PRECISION G$F$(LDG$F$),G$X$(LDG$X$,N),X(N),XBAR(*)

.. Local Scalars ..
DOUBLE PRECISION D$0$,D$4$,FLIN,FLINBAR,FQUAD,HXYHYX,V,VBAR,VD,
+ VBAR,VL,VBAR,VR,VRBAR,VO,VOBAR
INTEGER G$P$1,1,1,K

.. Local Arrays ..
DOUBLE PRECISION G$FLIN$(G$P$MAX$\$),G$FQUAD$(G$P$MAX$\$)

.. External Subroutines ..
EXTERNAL DGEMV

IF (G$P$.GT.G$P$MAX$\$) THEN
   PRINT *, 'Parameter g$p is greater than g$pmx.'
   STOP
END IF
HXY = HX/HY
HYX = HY/HX

C  Computation of the quadratic part of the function.
FQUAD = 0.0
DO $I/$ = 1,G/$P/$
   G/$FQUAD/($I/$) = 0.0d0
END DO
DO $I/$ = 1,XBAR/SIZE
   XBAR($I/$) = 0.0
END DO
DO 99998 J = 0,NY
   DO 99999 $I/$ = 0,NX
      K = NX* ($J-1) + $I/$
      V = 0.0
      VR = 0.0
      VU = 0.0
      IF ($I/$.GE./1 .AND. $J/.GE./1/) V = X(K)
      IF ($I/$.LT./NX .AND. $J/.GT./0/) VR = X(K+1)
      IF ($I/$.GT./0 .AND. $J/.LT./NY/) VU = X(K+NX)
      FQUAD = FQUAD + HXY*VU**2 + HXY*VR**2
      D$0 = VR - V
      D$4 = VU - V
      FQUAD = FQUAD + HYX*D$0**2 + HXY*D$4**2
      VBAR = -2*HYX*D$4 - 2*HYX*D$0
      VUBAR = 2*HYX*D$4
      VRBAR = 2*HYX*D$0
      IF ($I/$.GE./1 .AND. $J/.GE./1/) XBAR(K) = XBAR(K) + VBAR
      IF ($I/$.LT./NX .AND. $J/.GT./0/) XBAR(K+1) = XBAR(K+1) + VRBAR
      IF ($I/$.GT./0 .AND. $J/.LT./NY/) XBAR(K+NX) = XBAR(K+NX) + VUBAR
   99999 CONTINUE
   20 CONTINUE
99998 CONTINUE
C  c xbar is the vector of partial derivatives of the contribution c to fquad with respect to x. Since fquad was zero before this c loop, the derivative d$fquad = g$x * xbar.
C
CALL DGMV('n',G$P$,XBAR/SIZE,1.0d0,G$X$,LDX$G$X,XBAR,1,1.0d0,G$FQUAD, +
   1)
DO $I/$ = 1,XBAR/SIZE
   XBAR($I/$) = 0.0
END DO
DO 99996 J = 1,NY + 1
   DO 99997 $I/$ = 1,NX + 1
      K = NX* ($J-1) + $I/$
   99997 CONTINUE
   99996 CONTINUE

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V = 0.0
VL = 0.0
VD = 0.0
IF (I.LE.NX .AND. J.LE.NY) V = X(K)
IF (I.GT.1 .AND. J.LE.NY) VL = X(K-1)
IF (I.LE.NX .AND. J.GT.1) VD = X(K-NX)
C
fquad = fquad + hyx *(vl - v) ** 2 + hxy *(wd - v) ** 2
D$0 = VL - V
D$4 = VD - V
FQUAD = FQUAD + HYX*D$0*2 + HXY*D$4*2
VBAR = -2*HXY*D$4 - 2*HYX*D$0
VDBAR = 2*HXY*D$4
VLBAR = 2*HYX*D$0
IF (I.LE.NX .AND. J.LE.NY) XBAR(K) = XBAR(K) + VBAR
IF (I.GT.1 .AND. J.LE.NY) XBAR(K-1) = XBAR(K-1) + VLBAR
IF (I.LE.NX .AND. J.GT.1) XBAR(K-NX) = XBAR(K-NX) + VDBAR
99997 CONTINUE
99996 CONTINUE
C
xbar is the vector of partial derivatives of the contribution
C to fquad with respect to x. Since fquad was already initialized
C before this loop, the derivative d$fquad = d$fquad + g$x * xbar.
C
CALL DGEMV(’n’,G$P,XBARSIZE,1.0d0,G$X,LXS$X,XBAR,1,1.0d0,G$FQUAD,
+ 1)
C
Computation of the linear part of the function.
FLIN = 0.0
D0 G$I$3 = 1,G$P
G$FLIN(G$I$3) = 0.0d0
END D0
D0 I = 1,XBARSIZE
XBAR(I) = 0.0
END D0
D0 99995 K = 1,NX*NY
C
g$flin(k) = g$flin(k) + 1
FLINBAR = 1.0
XBAR(K) = XBAR(K) + FLINBAR
FLIN = FLIN + X(K)
99995 CONTINUE
C
again, d$fquad = d$fquad + g$x * xbar.
C
CALL DGEMV(’n’,G$P,XBARSIZE,1.0d0,G$X,LXS$X,XBAR,1,1.0d0,G$FLIN,
+ 1)
C
f = 0.25 * fquad - force * hx * hy * flin
DO G$13 = 1, G$3
    G$F(G$13) = 0.25 * G$FQUAD(G$13) - (FORCE*HX*HY)*G$FLIN(G$13)
END DO
F = 0.25*FQUAD - FORCE*HX*HY*FLIN
END
E Reverse Mode for the Whole Program

SUBROUTINE REV3(N,X,F,NX,HY,HX,HY,FORCE,XBAR,XBARSIZE)
C
C ADIFOR: runtime gradient index
C ADIFOR: translation time gradient index
C
***********
C This subroutine computes the function of the torsion problem.
C ***********
C The spacing parameters hx and hy are for a rectangle with
C nx points on the x-axis and ny points on the y-axis
C
C ADIFOR: gradient declarations
C .. Scalar Arguments ..
DOUBLE PRECISION F,FORCE,HX,HY
INTEGER N,NX,NY,XBARSIZE
C ..
C .. Array Arguments ..
DOUBLE PRECISION X(N),XBAR(*)
C ..
C .. Local Scalars ..
DOUBLE PRECISION D$0,D$4,FLIN,FLINBAR,FQUAD,HXY,HYX,T,V,VBAR,VD,
+ VDBAR,VL,VLBAR,VR,VRBAR,VU,VUBAR
INTEGER I,J,K
C
HXY = HX/HY
HYX = HY/HX
C
Computation of the quadratic part of the function.
FQUAD = 0.0
DO 1 I = 1,XBARSIZE
   XBAR(I) = 0.0
END DO
DO 99998 J = 0,NY
   DO 99999 I = 0,NX
      K = NX* (J-1) + I
      V = 0.0
      VR = 0.0
      VU = 0.0
      IF (I.GE.1 .AND. J.GE.1) V = X(K)
      IF (I.LT.NX .AND. J.GT.0) VR = X(K+1)
      IF (I.GT.0 .AND. J.LT.NY) VU = X(K+NX)
      fquad = fquad + hxy * (vr - v) ** 2 + hxy * (vu - v) ** 2
      D$0 = VR - V
      D$4 = VU - V
      FQUAD = FQUAD + HXY*D$0**2 + HXY*D$4**2
      VBAR = -2*HXY*D$4 - 2*HYX*D$0
C
**
VUBAR = 2*HXY*D$4
VRBAR = 2*HYX*D$0
IF (I.GE.1 .AND. J.GE.1) XBAR(K) = XBAR(K) + VBAR
IF (I.LT.NX .AND. J.GT.0) XBAR(K+1) = XBAR(K+1) + VRBAR
IF (I.GT.0 .AND. J.LT.NY) XBAR(K+NX) = XBAR(K+NX) + VUBAR
99999 CONTINUE
99998 CONTINUE

DO 99996 J = 1,NY + 1
   DO 99997 I = 1,NX + 1
      K = NX* (J-1) + I
      V = 0.0
      VL = 0.0
      VD = 0.0
      IF (I.LE.NX .AND. J.LE.NY) V = X(K)
      IF (I.GT.1 .AND. J.LE.NY) VL = X(K-1)
      IF (I.LE.NX .AND. J.GT.1) VD = X(K-NX)
      C fquad = fquad + hyx * (vl - v) ** 2 + hxy * (vd - v) ** 2
      D$0 = VL - V
      D$4 = VD - V
      FQUAD = FQUAD + HXY*D$0**2 + HXY*D$4**2
      VBAR = -2*HXY*D$4 - 2*HXY*D$0
      VDBAR = 2*HXY*D$4
      VLBAR = 2*HXY*D$0
      IF (I.LE.NX .AND. J.LE.NY) XBAR(K) = XBAR(K) + VBAR
      IF (I.GT.1 .AND. J.LE.NY) XBAR(K-1) = XBAR(K-1) + VLBAR
      IF (I.LE.NX .AND. J.GT.1) XBAR(K-NX) = XBAR(K-NX) + VDBAR
   CONTINUE
   CONTINUE
99997 CONTINUE
99996 CONTINUE
C Computation of the linear part of the function.
FLIN = 0.0
T = - (FORCE*HX*HY) cm all the flinbar(k)'s are equal to 1.0
FLINBAR = 1.0
DO 99995 K = 1,NX*NY
   XBAR(K) = (0.25*XBAR(K)) + (T*FLINBAR)
   FLIN = FLIN + X(K)
99995 CONTINUE
F = 0.25*FQUAD - FORCE*HX*HY*FLIN
END
F Derivative Code for Component Functions

SUBROUTINE REV0A(G$P$,N,X,G$X$,LDG$X$,F,NX,NY,HX,HY,FORCE,FQ,G$FQ$, + LDG$FQ$)

***********
* derivative of first loop -- current ADIFOR
***********

C Formal fq is active.
C Formal x is active.
C
C .. Parameters..
INTEGER G$P$MAX$
PARAMETER (G$P$MAX$=100)
C..
C .. Scalar Arguments..
DOUBLE PRECISION F,FORCE,HX,HY
INTEGER G$P$,LDG$FQ$,LDG$X$,N,NX,NY
C ..
C .. Array Arguments..
DOUBLE PRECISION FQ(*),G$FQ$(LDG$FQ$,*)G$X$(LDG$X$,N),X(N)
C ..
C .. Local Scalars..
DOUBLE PRECISION D$0$,D$0BAR$,D$3$,D$3BAR$,FQUAD,HXY,HYX,V,VR,VU
INTEGER FQK,G$I$',I,J,K
C..
C .. Local Arrays..
DOUBLE PRECISION G$V$(G$P$MAX$),G$VR$(G$P$MAX$),G$VU$(G$P$MAX$)
C..
IF (G$P$.GT.G$P$MAX$) THEN
    PRINT *, 'Parameter g$p is greater than g$pmax.'
    STOP
END IF
HXY = HX/HY
HYX = HY/HX
C Computation of the quadratic part of the function.
FQUAD = 0.0
DO 99998 J = 0,NY
    DO 99999 I = 0,NX
        K = NX* (J-1) + I
        V = 0.0
        DO 99990 G$IS$ = 1,G$PS$
            G$V$(G$IS$) = 0.0d0
    99990 CONTINUE
    VR = 0.0
    DO 99989 G$IS$ = 1,G$PS$
        G$VR$(G$IS$) = 0.0d0

99998 CONTINUE
CONTINUE
VU = 0.0
DO 99988 GIS = 1,GS$GIS
     G$VU(GIS) = 0.0d0
CONTINUE

IF (I.GE.1 .AND. J.GE.1) THEN
   V = x(k)
   DO 99987 GIS = 1,GS$GIS
       G$VU(GIS) = G$X(GIS,k)
CONTINUE
   V = X(k)
END IF
IF (I.LT.NX .AND. J.GT.0) THEN
   VR = x(k + 1)
   DO 99986 GIS = 1,GS$GIS
       G$VR(GIS) = G$X(GIS,k+1)
CONTINUE
   VR = X(k+1)
END IF
IF (I.GT.0 .AND. J.LT.NY) THEN
   VU = x(k + nx)
   DO 99985 GIS = 1,GS$GIS
       G$VU(GIS) = G$X(GIS,k+nx)
CONTINUE
   VU = X(k+nx)
END IF
FQuad = fQuad + hxy*(VR-v)**2 + hxy*(VU-v)**2
FQK = (J* (NX+1)) + I + 1

FQ(FQK) = hxy * (VR - v) ** 2 + hxy * (VU - v) ** 2
D$0 = VR - V
D$3 = VU - V
D$3BAR = hxy* (2*D$3)
D$0BAR = hxy* (2*D$0)
DO 99984 GIS = 1,GS$GIS
     G$FQ(GIS,FQK) = D$0BAR*G$VR(GIS) +
     (-D$3BAR+ (-D$0BAR))*G$V(GIS) +
     D$3BAR*G$VU(GIS)
CONTINUE
FQ(FQK) = HXY*D$0**2 + HXY*D$3**2

CONTINUE
CONTINUE
CONTINUE
END
SUBROUTINE REV80(B(G$/P/$,N,X,G$/X$/X,F,NX,MY,HY,FORCE,FQQ,G$/FQQ$/,
+ LDG$/FQQ$/)

************
* deriv for second loop, current ADIFOR
************
C
C Formal fqq is active.
C Formal x is active.
C
C .. Parameters ..
INTEGER G$/PMAX/$
PARAMETER (G$/PMAX/$=100)
C ..
C .. Scalar Arguments ..
DOUBLE PRECISION F,FORCE,HX,H Y
INTEGER G$/P/$,LDG$/FQQ$/LDG$/X/$,NX,N Y
C ..
C .. Array Arguments ..
DOUBLE PRECISION FQQ(*/,),G$/FQQ/(LDG$/FQQ$/),G$/X /(LDG$/X$/),X(N)
C ..
C .. Local Scalars ..
DOUBLE PRECISION D$/0$/,D$/0$BAR/,D$/3$/,$D$/3$BAR/,HXY,H YX,V,VD,VL
INTEGER FQK,G$/I$/,$I$/,I/,J/, K
C ..
C .. Local Arrays ..
DOUBLE PRECISION G$/V/(G$/PMAX/$/),G$/VD /(G$/PMAX$/),G$/VL/(G$/PMAX$/)
C ..
IF (G$/P/$.GT./G$/PMAX/$) THEN
PRINT /*/,/'Parameter g$/p is greater than g$/pmax/.'
STOP
END IF
HXY = HX/HY
HYX = HY/HX
DO 99998 J = 1,NY + 1
   DO 99999 I = 1,NX + 1
       K = NX* (J-1) + I
       V = 0.0
       DO 99990 G$/I$/ = 1,G$/P$/
              G$/V$(G$/I$/) = 0.0d0
         99990 CONTINUE
   VL = 0.0
   DO 99989 G$/I$/ = 1,G$/P$/
          G$/VL$(G$/I$/) = 0.0d0
99989 CONTINUE
   VD = 0.0
   DO 99988 G$/I$/ = 1,G$/P$/
   99988 CONTINUE

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G$VD(G$S$I$) = 0.0d0
CONTINUE
IF (I.LE.NX .AND. J.LE.NY) THEN
  V = x(k)
  DO 99987 G$S$I$ = 1,G$S$P$
  G$SV(G$S$I$) = G$X(G$S$I$,K)
  CONTINUE
  V = X(K)
END IF
IF (I.GT.1 .AND. J.LE.NY) THEN
  Vl = x(k - 1)
  DO 99986 G$S$I$ = 1,G$S$P$
  G$SVL(G$S$I$) = G$X(G$S$I$,K -1)
  CONTINUE
  VL = X(K-1)
END IF
IF (I.LE.NX .AND. J.GT.1) THEN
  Vd = x(k - nx)
  DO 99985 G$S$I$ = 1,G$S$P$
  G$SVD(G$S$I$) = G$X(G$S$I$,K-NX)
  CONTINUE
  VD = X(K-NX)
END IF
C
  fquad = fquad + hyx*(vl-v)**2 + hxy*(vd-v)**2
  FQK = ((J-1)* (NX+1)) + I
C
  fqq(fqk) = hyx * (vl - v) ** 2 + hxy * (vd - v) ** 2
  D$0 = VL - V
  D$3 = VD - V
  D$3BAR = HXY* (2*D$3)
  D$0BAR = HYX* (2*D$0)
  DO 99984 G$S$I$ = 1,G$S$P$
    G$S$FQQ(G$S$I$,FQK) = D$0BAR*G$SVL(G$S$I$) +
    (-D$3BAR* (-D$0BAR)) * G$SV(G$S$I$) +
    D$3BAR*G$SVD(G$S$I$)
  CONTINUE
  FQQ(FQK) = HYX*D$0**2 + HXY*D$3**2
CONTINUE
CONTINUE
CONTINUE
END
SUBROUTINE REV1C(G$P$,N,X,G$X$,LDG$X$,F,NX,MY,HX,HY,FORCE,FP,RGFP, 
   + LDG$FP$)

**************
* third loop, basic block reverse mode
**************

C     .. Scalar Arguments ..
DOUBLE PRECISION F,FORCE,HX,HY
INTEGER G$P$,LDG$FP$,LDG$X$,N,NX,MY
C     ..
C     .. Array Arguments ..
DOUBLE PRECISION FP(*),G$X$(LDG$X$,*),RGFP(LDG$FP$,*),X(N)
C     ..
C     .. Local Scalars ..
DOUBLE PRECISION FPBAR,HXY,HX
INTEGER I,K
C
   HXY = HX/HY
   HYX = HY/HX

C Computation of the linear part of the function.

DO 50 K = 1,NX*NY
   FPBAR = 1.0
   RGFP(I,K) = 0.0
   DO I = 1,G$P$
      RGFP(I,K) = RGFP(I,K) + FPBAR*G$X$(I,K)
   END DO
   FP(K) = X(K)
50 CONTINUE

END
G Enhanced Derivative Code for Component Functions

SUBROUTINE REV1A(G$P,N,X,G$X,LDG$X,F,NX,NY,HX,HY,FORCE,FQ,RGFQ,
+ LDG$FQ)

********
* first loop contribution, basic block reverse mode
********
C
C ADIFOR: runtime gradient index
C ADIFOR: translation time gradient index
C ADIFOR: gradient iteration index
C
C The spacing parameters hx and hy are for a rectangle with
C nx points on the x-axis and ny points on the y-axis
C
C ADIFOR: gradient declarations
C
.. Parameters ..
INTEGER G$PMAX$ PARAMETER (G$PMAX$=4900)
C
.. Scalar Arguments ..
DOUBLE PRECISION F,FORCE,HX,HY
INTEGER G$P$,LDG$FQ,LDG$X,N,NX,NY
C
.. Array Arguments ..
DOUBLE PRECISION FQ(*),G$X(LDG$X,N),RGFQ(LDG$FQ,*),X(N)
C
.. Local Scalars ..
DOUBLE PRECISION D$0,D$4,QUAD,HXY,V,BAR,V,R,VBAR,VU,VUBAR,
+ X1BAR,X2BAR,XHXBAR
INTEGER FQK,I,J,K,PP
C
.. IF (G$P$.GT.G$PMAX$) THEN
   PRINT *, 'Parameter g$p is greater than g$pmx.'
   STOP
END IF
HXY = HX/HY
HYX = HY/HX
C
Computation of the quadratic part of the function.a
C the following is not needed
C do g$i$ = 1, g$P$
C   do j = 1, ((nx+1)*(ny+1))
C      RGFQ(g$i$,j) = 0.0d0
C   enddo
C enddo
C
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FQUAD = 0.0
DO 20 J = 0,NY
   DO 10 I = 0,NX
      K = NX*(J-1) + I
      V = 0.0
      VR = 0.0
      VU = 0.0
      IF (I.GE.1 .AND. J.GE.1) V = X(K)
      IF (I.LT.NX .AND. J.GT.0) VR = X(K+1)
      IF (I.GT.0 .AND. J.LT.NY) VU = X(K+NX)
      FQK = (J*(NX+1)) + I + 1
      FQ(FQK) = HYX*(VR-V)**2 + HXY*(VU-V)**2
      D$0 = VR - V
      D$4 = VU - V
      FQ(FQK) = HYX*D$0**2 + HXY*D$4**2
      VBAR = -2*HYX*D$4 - 2*HYX*D$0
      VUBAR = 2*HYX*D$4
      VRBAR = 2*HYX*D$0
      XKBAR = 0.0
      XK1BAR = 0.0
      XKNXBAR = 0.0
      IF (I.GE.1 .AND. J.GE.1) XKBAR = VBAR
      IF (I.LT.NX .AND. J.GT.0) XK1BAR = VRBAR
      IF (I.GT.0 .AND. J.LT.NY) XKNXBAR = VUBAR
      DO PP = 1,G$P$
         RGFQ(PP,FQK) = XKBAR*G$X(PP,K) + XK1BAR*G$X(PP,K+1) +
                        XKNXBAR*G$X(PP,K+NX)
      END DO
   10 CONTINUE
20 CONTINUE
END
SUBROUTINE REV1B(G$P$,N,X,G$S$,X,F,NX,NY,HX,HY,FORCE,FQQ,RGFQQ, 
+ LDG$FQQ$)

************
* second loop, basic block reverse mode
************

C ADIFOR: runtime gradient index
C ADIFOR: translation time gradient index
C ADIFOR: gradient iteration index
C
C The spacing parameters hx and hy are for a rectangle with
C nx points on the x-axis and ny points on the y-axis
C
C ADIFOR: gradient declarations
C
.. Parameters..
INTEGER G$PMAX$
PARAMETER (G$PMAX$=4900)
C ..
C .. Scalar Arguments ..
DOUBLE PRECISION F,FORCE,HX,HY
INTEGER G$P$,LDG$FQQ$,LDG$X$,N,NX,NY
C ..
C .. Array Arguments ..
DOUBLE PRECISION FQQ(*),G$S$(LDG$X$,N),RGFQQ(LDG$FQQ$,*)x(N)
C ..
C .. Local Scalars ..
DOUBLE PRECISION D0,D$4$,FQUAD,HXY,V,VBAR,WD,VDBAR,VL,VLBAR, 
+ XKBAR,XKXBAR
INTEGER FQK,I,J,K,FP
C ..
IF (G$P$.GT.G$PMAX$) THEN
  PRINT *, 'Parameter g$P$ is greater than g$pmax'.
STOP
END IF
HXY = HX/HY
HYX = HY/HX
C Computation of the quadratic part of the function.
C the following is NOT needed
C do g$i$ = 1, g$P$
C     do j= 1, ((nx+1)*(ny+1))
C       rgfQQ(g$i$,j) = 0.0do
C enddo
C enddo
FQUAD = 0.0
DO 40 J = 1,NY + 1
  DO 40 I = 1,NX + 1
40    FQUAD = FQUAD + FQQ(I,J)

D0 30 I = 1,NX + 1
   K = NX* (J-1) + I
   V = 0.0
   VL = 0.0
   VD = 0.0
   IF (I.LE.NX .AND. J.LE.NY) V = X(K)
   IF (I.GT.1 .AND. J.LE.NY) VL = X(K-1)
   IF (I.LE.NX .AND. J.GT.1) VD = X(K-NX)
   cm
   fquad = fquad + hyx*(vl-v)**2 + hxy*(vd-v)**2
   FQK = ((J-1)* (NX+1)) + I
   FQQ(FQK) = HYX* (VL-V)**2 + HXY* (VD-V)**2
   D$0 = VL - V
   D$4 = VD - V
   FQQ(FQK) = HYX*D$0**2 + HXY*D$4**2
   VBAR = -2*HXY*D$4 - 2*HYX*D$0
   VLBAR = 2*HYX*D$0
   VDBAR = 2*HXY*D$4
   XKBAR = 0.0
   XK1BAR = 0.0
   XKNXBAR = 0.0
   IF (I.LE.NX .AND. J.LE.NY) XKBAR = VBAR
   IF (I.GT.1 .AND. J.LE.NY) XK1BAR = VLBAR
   IF (I.LE.NX .AND. J.GT.1) XKNXBAR = VDBAR
   cm
   D0 PP = 1,G$P$
   RGFQQ(PP,FQK) = XKBAR*G$X(PP,K) + XK1BAR*G$X(PP,K-1) +
                     XKNXBAR*G$X(PP,K-NX)
   END D0

30 CONTINUE
40 CONTINUE
END
SUBROUTINE REVIC(G$P$,N,X,G$X$,LDG$X$,F,NX,MY,HY,FORCE,FP,RGFP, + 
LDG$FP$)

***************
* third loop, basic block reverse mode 
***************

C .. Scalar Arguments ..
DOUBLE PRECISION F,FORCE,HX,HY
INTEGER G$P$,LDG$FP$,LDG$X$,N,NX,MY
C ..
C .. Array Arguments ..
DOUBLE PRECISION FP(*),G$X$(LDG$X$,*),RGFP(LDG$FP$,*) X(N)
C ..
C .. Local Scalars ..
DOUBLE PRECISION FPBAR,HXY,HYX
INTEGER I,K
C ..
HXY = HX/HY
HYX = HY/HX
C Computation of the linear part of the function.
DO 50 K = 1,NX*NY
   FPBAR = 1.0
   RGFP(I,K) = 0.0
   DO I = 1,G$P$
      RGFP(I,K) = RGFP(I,K) + FPBAR*G$X$(I,K)
   END DO
   FP(K) = X(K)
50 CONTINUE

END